where

$$C_{L_\epsilon}(t) = \frac{\beta^2}{k}\bar{C}_{L_\epsilon}(t) \qquad (27)$$

A similar analysis for the pitching moment C_m results in

$$C_m(t) = C_{m_0} + \epsilon(0)C_{m_\epsilon}(t) + \int_0^t C_{m_\epsilon}(\tau)\frac{d\epsilon(t-\tau)}{dt}d\tau \qquad (28)$$

where

$$C_{m_\epsilon}(t) = 1/\epsilon_0[C_m^I(t) - C_{m_0}] \qquad (29)$$

and $C_m^I(t)$ is the indicial response for the pitching moment.

Equations (26) and (28) are the same equations used by Ballhaus and Goorjian.[1] Hence, while the shock motion must be taken into account explicitly when the pressure coefficient is to be calculated, it need not be for the calculation of integrated parameters such as C_L and C_m. The reason for this is that the shock movement is implicitly taken into account in the computation of the indicial responses $C_{L_\epsilon}(\epsilon)$ and $C_{m_\epsilon}(t)$.

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Nonlinear Formulation for Low-Frequency Transonic Flow

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Introduction

WE will consider the two-dimensional unsteady transonic flow past a thin airfoil executing small-amplitude harmonic oscillations. The usual approach[1] solves a (time) Fourier-transformed problem linearized about a prescribed steady flow. The mean flow is obtained (once) from a Murman-Cole method, and its solution determines the variable coefficients of a linearized, frequency-dependent, mixed-type problem. This linearization, however, is not uniformly valid in time: the *nonharmonic* part of the total flow *must* change for different disturbance frequencies and amplitudes because the problem is nonlinear. The mean flow cannot be assumed known for all time; the "ϵ expansion" (i.e., the straightforward linearization) therefore breaks down because it does not describe the "back-interaction" mechanism that arises from the nonlinear harmonic interplay required on physical grounds.

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This defect is remedied by equating *not* coefficients of like powers of a small-amplitude measure (in determining a sequential ordering of problems), but coefficients of like *harmonics*. The resulting equations describe the feedback phenomenon, and if they can be solved exactly, no physical approximation will be involved. However, for simplicity, we will consider the effect of the primary harmonic only and use the small-disturbance formulation (there is no conceptual difficulty in writing down more terms, however). The back-interaction effect, of course, will be important for low-frequency problems where shock excursions are generally large, and may have a significant effect on mean shock jumps and location. For high frequencies, the effect is probably less important, and a linearized approach should do. This Note shows how flutter criteria depend on both frequency and amplitude boundaries in a physically self-consistent way.

Analysis

Let M_∞ be the subsonic freestream Mach number, U the freestream speed, C the chord, δ the thickness ratio, and γ the ratio of specific heats. We normalize the streamwise coordinate x_0, the normal coordinate y_0, and the time t_0 by defining nondimensional variables by $x = x_0/C$, $y = [(1+\gamma)\delta M_\infty^2]^{1/3} y_0/C$, and $t = [(1+\gamma)\delta M_\infty^2]^{1/3} Ut_0/M_\infty^2 C$, and introduce a nondimensional disturbance potential φ by expanding the total dimensional potential in the form

$$\Phi \cong UCx + \frac{\delta^{2/3} UC}{[(1+\gamma)M_\infty^2]^{1/3}}\varphi(x,y,t)$$

Let us also introduce a reduced frequency $k = \omega C/U$, where ω is the frequency of the oscillation, and a nondimensional frequency $\Omega = M_\infty^2 k/[(1+\gamma)\delta M_\infty^2]^{1/3}$. Substitution into the full potential equation leads to the small-perturbation equation

$$(K^* - \varphi_x)\varphi_{xx} + \varphi_{yy} = 2\varphi_{xt} + (k/\Omega)\varphi_{tt} \qquad (1)$$

where the transonic similarity parameter $K^* = (1-M_\infty^2)/[(1+\gamma)\delta M_\infty^2]^{2/3}$. Let $y = g_{u,\ell}(x,t)$ represent upper and lower wing surfaces. Then, Eq. (1) is solved along with the following linearized boundary conditions:

$$\varphi_y(0<x<1;\quad y=\pm 0) = \left(\frac{\partial}{\partial x} + \frac{K}{\Omega}\frac{\partial}{\partial t}\right)g_{u,\ell}(x,t) \qquad (2)$$

$$\left(\varphi_x + \frac{k}{\Omega}\varphi_t\right)_{\substack{x>1\\ y=0+}} = \left(\varphi_x + \frac{k}{\Omega}\varphi_t\right)_{\substack{x>1\\ y=0-}} \qquad (3)$$

$$\varphi_x^2 + \varphi_y^2 \to 0 \quad \text{as } x^2+y^2 \to \infty \qquad (4)$$

We assume that the unsteady motion is a small perturbation on the steady flow, and characterize it by a small nondimensional displacement ϵ and the reduced frequency. In view of the preceding discussion, we expand both the airfoil motion and the disturbance potential in *real* harmonic series,

$$g(x,t) = g_0(x) + \tfrac{1}{2}\epsilon\{g_1(x)e^{i\Omega t} + \bar{g}_1 e^{-i\Omega t}\} + \ldots \qquad (5)$$

$$\varphi(x,y,t) = \varphi_0(x,y) + \tfrac{1}{2}\epsilon\{\varphi_1(x,y)e^{i\Omega t} + \bar{\varphi}_1 e^{-i\Omega t}\} + \ldots \qquad (6)$$

bars denoting complex conjugates (higher harmonics are related to higher-order amplitude effects that can be consistently neglected). Substitution in Eqs. (1-3) leads to the mean flow formulation

$$\frac{\partial}{\partial x}\left\{K^*\varphi_{0_x} - \frac{1}{2}\varphi_{0_x}^2 - \frac{1}{4}\epsilon^2|\varphi_{1_x}|^2\right\} + \frac{\partial}{\partial y}\{\varphi_{0_y}\} = 0 \qquad (7)$$

$$\varphi_{0_y}(0<x<1;\quad y=\pm 0) = \partial g_{0_{u,\ell}}(x)/\partial x \qquad (8)$$

$$\varphi_{0_x}(x>l;\quad y=0+)=\varphi_{0_x}(x>l;\quad y=0-) \tag{9}$$

for $\varphi_0(x,y)$, on equating coefficients of the zeroth harmonic, and on examining the first harmonic, to the following φ_l formulation:

$$(K^*-\varphi_{0_x})\varphi_{l_{xx}}+\varphi_{l_{yy}}-(\varphi_{0_{xx}}+2i\Omega)\varphi_{l_x}+k\Omega\varphi_l=0 \tag{10}$$

$$\varphi_{l_y}(0<x<l;\quad y=\pm 0)=(g_{l_x}+ikg_l)_{u,\ell} \tag{11}$$

$$(\varphi_x+ik\varphi_l)_{\substack{x>l\\y=0+}}=(\varphi_{l_x}+ik\varphi_l)_{\substack{x>l\\y=0-}} \tag{12}$$

Regularity conditions, of course, are imposed on both of the preceding formulations (the inclusion of the φ_{tt} term renders the theory applicable to reduced frequencies of order unity). If the ϵ^2 term in Eq. (7) is dropped, φ_0 decouples from φ_l, and the governing unsteady equations reduce to the usual linearized equations. These however, do not account for modifications to the mean flow (induced by the unsteady, nonlinear back-interaction that must become important over large time scales). To retain the nonlinear coupling requires a simultaneous solution for φ_0 and φ_l, which will, in practice, be a numerical one.

The nonlinear formulation must be completed by proper specification of shock conditions in the event that discontinuities form. It is essential to use the physically correct conservation form of Eq. (1). This is achieved by recognizing Eq. (1) as a statement of mass conservation[2] [the time rate of change of density, for example, contributes a $(k/\Omega)\varphi_{tt}$ term and one φ_{xt} term, while the remaining φ_{xt} term arises from the streamwise mass flux term]. When viewed in this manner, the required conservation form is

$$\frac{\partial}{\partial t}\left(-\varphi_x-\frac{k}{\Omega}\varphi_t\right)+\frac{\partial}{\partial x}\left(K^*\varphi_x-\frac{1}{2}\varphi_x^2-\varphi_t\right)+\frac{\partial}{\partial y}(\varphi_y)=0 \tag{13}$$

We will also invoke irrotationality, that is, $v_x-u_y=0$ (u and v are streamwise and stream-normal velocity perturbations, respectively). Now let $\psi(x,y,t)=x-x_s(y,t)=0$ describe the instantaneous shock surface $x=x_s(y,t)$. Corresponding to Eq. (13), we have

$$[-\varphi_x-(k/\Omega)\varphi_t]\psi_t+[K^*\varphi_x-\tfrac{1}{2}\varphi_x^2-\varphi_t]\psi_x+[\varphi_y]\psi_y=0 \tag{14}$$

where [ ] denotes the jump of the enclosed quantity across x_s, while from irrotationality, we have

$$[\varphi_y]\psi_x-[\varphi_x]\psi_y=0 \tag{15}$$

Next, expand $x_s(y,t)$ in the form

$$x_s(y,t)=x_M(y)+\tfrac{1}{2}\epsilon(e^{i\Omega t}f(y)+e^{-i\Omega t}\bar{f}(y))+\ldots \tag{16}$$

where the subscript M denotes the unknown mean shock location. Now, $\varphi(x_s)\cong\varphi(x_M)+(x_s-x_M)\varphi_x(x_M)$ can be rewritten by replacing x_s-x_M using Eq. (16) and then introducing Eq. (6). Substitution in $[\varphi]=0$, which is implied by Eq. (15), leads to a result that expresses jumps in φ_0 and φ_l and their derivatives about the mean shock location x_M. Equating coefficients of the zeroth harmonic leads to

$$[\varphi_0]_{x_M}+\tfrac{1}{4}\epsilon^2\{f[\bar{\varphi}_{l_x}]_{x_M}+\bar{f}[\varphi_{l_x}]_{x_M}\}=0 \tag{17}$$

where the jump is evaluated at x_M, while coefficients of the first harmonic lead to

$$[\varphi_l]_{x_M}+f[\varphi_{0_x}]_{x_M}=0 \tag{18}$$

Next we expand out Eq. (14), using Eq. (16). Again we express the jump conditions for both mean and unsteady problems about the mean shock location $x=x_M(y)$. Equation (16) is first substituted into Eq. (14); then, we expand $\varphi(x_s)$ in a Taylor series about $x=x_M$, and introduce Eq. (6) into the resulting equation. For simplicity, the resulting equation is truncated to retain only those nonlinear effects due to the primary harmonic. Thus, the jump condition for the mean flow becomes

$$[K^*\varphi_{0_x}-\tfrac{1}{2}\varphi_{0_x}^2-\varphi_{0_y}x_M']_{x_M}$$

$$=\tfrac{1}{4}\epsilon^2\begin{bmatrix}|\varphi_{l_x}|^2+|f|^2\varphi_{0_{xx}}^2+\varphi_{0_{xx}}(\varphi_{l_x}\bar{f}+\bar{\varphi}_{l_x}f)\\+\varphi_{0_x}(\varphi_{l_{xx}}\bar{f}+f\bar{\varphi}_{l_{xx}})+\varphi_{0_{yx}}(f\bar{f}'+f'\bar{f})\\+(\varphi_{l_y}\bar{f}'+f'\bar{\varphi}_{l_y})+x_M'(\bar{f}\varphi_{l_{yx}}+f\bar{\varphi}_{l_{yx}})\\-2i\Omega(f\bar{\varphi}_{l_x}-\bar{f}\varphi_{l_x})-K^*(\bar{f}\varphi_{l_{xx}}+f\bar{\varphi}_{l_{xx}})\\-\Omega k(\varphi_l\bar{f}+\bar{\varphi}_l f)\end{bmatrix}_{x_M} \tag{19}$$

while for the flow related to the primary harmonic, we have

$$[(K^*-\varphi_{0_x})(\varphi_{l_x}+f\varphi_{0_{xx}})-i\Omega(\varphi_l-\varphi_{0_x}f)-\varphi_{0_y}f'$$
$$-x_M'(\varphi_{l_y}+f\varphi_{0_{yx}})]_{x_M}=0 \tag{20}$$

For convenience, the f's and x_M's are shown within the square brackets. Finally, the jump conditions corresponding to Eq. (15) can be shown to be

$$[\varphi_{0_y}+\varphi_{0_x}x_M']_{x_M}=-\tfrac{1}{4}\epsilon^2\begin{bmatrix}\varphi_{0_{xx}}(f\bar{f}'+\bar{f}f')\\+(\varphi_{l_x}\bar{f}'+f'\bar{\varphi}_{l_x})\\+x_M'(\bar{f}\varphi_{l_{xx}}+f\bar{\varphi}_{l_{xx}})\end{bmatrix}_{x_M} \tag{21}$$

$$[\varphi_{l_y}+\varphi_{0_x}f'+x_M'f\varphi_{0_{xx}}+x_M'\varphi_{l_x}]_{x_M}=0 \tag{22}$$

This completes the nonlinear formulation. The preceding equations must, in general, be solved iteratively.

Discussion and Summary

We suggest a basic numerical approach that consists in solving and updating the mean and disturbance flows iteratively. For example, one relaxation cycle using Eqs. (7-9) can be followed by one using Eqs. (10-12), taking proper account of conservation form; the resulting φ_l can be used in Eqs. (7-9) to update φ_0, and so on, with the converged solution used to calculate the shock motion. It is not clear whether or not this type of iteration contains any implied assumptions on the weakness of the back-interaction, but algorithms developed along these lines require little more than simple modifications to existing baseline codes. In the long run, though, direct ADI time-integration schemes may be more economical. But if harmonic analyses are required (as in flutter research), the present work may be more useful insofar as indicating the extent to which purely linearized approaches, by themselves, may suffice. When the nonlinear coupling can be safely disregarded, and what the qualitative features of nonlinearity are (for example, with regard to shock jumps and location, the resolution of steady and unsteady forces, etc.) are objectives of some numerical studies currently under way at Boeing.

Acknowledgment

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Drop Breakup and Liquid Jet Penetration

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THE transient response of a liquid drop to a suddenly applied aerodynamic flowfield has been studied extensively in shock tubes by establishing a falling column of drops in the driven section and observing their behavior after shock passage. Reference 1 provides a summary of such work and lists relevant references. The behavior of a liquid jet steadily injected into a gas stream also has received extensive study in supersonic wind tunnels, where the jet is injected normal to the gas flow from the side wall or from a flat plate aligned with the stream. Reference 2 presents the results of such experiments and a list of references. The purpose of this Note is to show the phenomenological and quantitative similarity of drop shattering and liquid jet penetration.

Experimental observations of drop behavior after shock passage have included the time t_d for the drop to shatter and its drag-induced displacement x from the initial location. These quantities generally have been nondimensionalized by the diameter D and density ρ_d of the drop and by the air density ρ_2 and speed U_2 behind the shock wave and relative to the undisturbed drops, resulting in the dimensionless quantities

$$X = x/D \qquad (1)$$

$$T_d = t_d U_2 \sqrt{\rho_2/\rho_d}/D \qquad (2)$$

The dimensionless breakup time defined in this way has been shown to depend primarily, but weakly, on the Weber number

$$We = D\rho_2 U_2^2/\sigma \qquad (3)$$

where σ is the drop surface tension.[1] Specifically, T_d varies as $We^{-1/4}$, a fact that indicates that a drop is shattered by the growth of unstable Rayleigh-Taylor waves on its front surface.[1,3] For Weber numbers greater than about 500, T_d is of order 4.[1] The downstream displacement of the drop commonly is correlated for engineering purposes in the form

$$X = AT^2 \qquad (4)$$

where A varies from 0.5 to 1.4 for Weber numbers above 300 and depends on the Weber number and the time interval observed after shock passage.[1,4]

Turning to the phenomenon of liquid jet penetration into a gas flow, the experimental observations include the height h of jet penetration and the coordinates x, y of the jet.[2,5] The penetration commonly is nondimensionalized and correlated as

$$hM/D = K\sqrt{p_j^0/p} \qquad (5)$$

where D is the jet initial (or orifice) diameter, M and p are the freestream Mach number and static pressure, and p_j^0 is the jet plenum pressure. Since the liquid jet is usually underexpanded, Eq. (5) also can be written as

$$h/D = K\sqrt{\gamma\rho_j U_j^2/2\rho U^2} \qquad (6)$$

where $\rho_j U_j^2$ and ρU^2 are the jet and airstream dynamic pressures, respectively, and γ is the specific heat ratio of the air. Typical values of K were about 7 in experiments where the Weber number was generally above 500.[2,5]

The breakup of the liquid jet is similar to the breakup of a drop in that, upon emerging from the jet orifice, an element of the jet is, like a drop overtaken by a shock, suddenly subjected to aerodynamic forces tending to accelerate and shatter it. Assuming that the jet element moves normal to the air velocity at constant speed U_j, the distance h that the jet penetrates into the flow is the product of the time t_j required to shatter the jet element times the speed of the jet, that is, $h = t_j U_j$. From this relation and from Eq. (6), we can derive the dimensionless time for jet breakup:

$$T_j = t_j U\sqrt{\rho/\rho_j}/D = 5.85 \quad (\text{for } \gamma = 1.4,\ K = 7) \qquad (7)$$

Comparison of this result with Eq. (2) shows that the forms of the nondimensional times are identical. Moreover, the values of the dimensionless times T_d and T_j are quite close, about 4 and 6, respectively.

We also can compare the downstream displacement of the jet element with that of the drop, since both motions result from aerodynamic drag. Values of the downstream displacement of the jet element as a function of its time after injection y/U_j, where x and y are the coordinates of the jet measured from the orifice parallel and normal to the freestream velocity, respectively, are available in Ref. 5. A plot of these data in the nondimensional form of $X = x/D$ as a function of $T = (y/U_j)(U/D)\sqrt{\rho/\rho_j}$ is shown in Fig. 1, together with a parabolic least-squares regression:

$$X = 0.8T^2, \quad T \leq 6 \qquad (8)$$

Comparison of this result with Eq. (4) again indicates both the similarity of the dimensionless groups and correlation and the agreement of the empirical coefficients. This agreement results not only from the physical similarity of the jet and drop shattering phenomena, but also from the fact that in

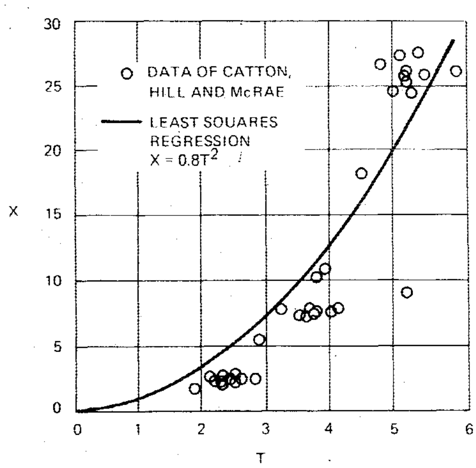

Fig. 1 Dimensionless downstream displacement of a liquid jet.

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